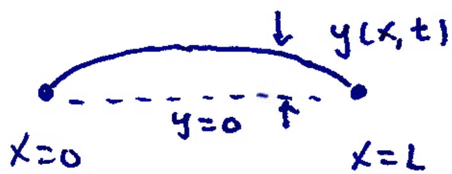
Wave Equation

models a vibrating string

string length L, held tight, ends fixed

displacement from natural position is $y(x,t)$



intuitively, the string does not "want" to be displaced from natural position (like a spring)

so, if displaced like  $y_{xx} < 0$, the string will have a downward restoring acceleration → $y_{tt} < 0$

↳ negative

similarly, if displaced like  $y_{xx} > 0$, string supplies an

upward restoring acceleration $\rightarrow y_{tt} > 0$

$\rightarrow$ acceleration is proportional to concavity (same sign)

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

<u>1-D Wave Equation</u>

or $$y_{tt} = a^2 y_{xx}$$

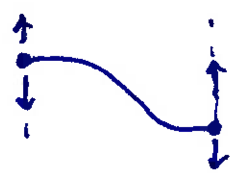
a^2 determines wave/vibration speed

a^2 in context of ~~sp~~ string

is $\dfrac{\text{tension}}{\text{density}}$

Boundary condition: $y(0,t) = y(L,t) = 0$ ends fixed at zero (Dirichlet)

$y_x(0,t) = y_x(L,t) = 0$ ends fixed horizontally (Neumann)
(can move up / down)



Initial conditions : two needed since time derivative is order 2

$y(x,0) = f(x)$ initial displacement (plucking)

$y_t(x,0) = g(x)$ initial velocity (strumming)

today, we will solve the case w/ ends fixed at 0 and initial displacement only ("Problem A")

$$y_{tt} = a^2 y_{xx} \qquad 0 < x < L, \quad t > 0$$

$y(0,t) = y(L,t) = 0$ ends fixed at 0

$y_t(x,0) = 0$ no initial velocity

$y(x,0) = f(x)$ initial displacement

we will again use the method of separation of variables

$$y(x,t) = X(x)\,T(t)$$

$$y_{tt} = X\,T'' \qquad y_{xx} = X''\,T$$

wave eq. becomes

$$X T'' = a^2 X'' T$$

$$\frac{X''}{X} = \frac{T''}{a^2 T} = \text{separation constant} = -\lambda$$ just like w/ heat eq.

we get two ODEs out of that:

$$X'' + \lambda X = 0$$

$$T'' + a^2 \lambda T = 0$$

BC: $y(0,t) = 0 \rightarrow X(0)T(t) = 0 \rightarrow X(0) = 0$

$y(L,t) = 0 \rightarrow X(L)T(t) = 0 \rightarrow X(L) = 0$

X solution is something we've seen before in heat eq.

⋮

$$\boxed{\lambda_n = \frac{n^2 \pi^2}{L^2}}$$ eigenvalues

$$\boxed{X_n = \sin\left(\frac{n\pi x}{L}\right)}$$ eigenfunctions

$n = 1, 2, 3, \ldots$

$$T'' + a^2 \lambda T = 0$$

$$T'' + \frac{a^2 n^2 \pi^2}{L^2} T = 0$$

$$T(t) = A \cos\left(\frac{an\pi}{L} t\right) + B \sin\left(\frac{an\pi}{L} t\right)$$

IC: $y_t(x, 0) = 0$ no initial velocity

$X(x) T'(0) = 0 \rightarrow T'(0) = 0$

⋮

$B = 0$

$$\boxed{T_n = \cos\left(\frac{n\pi a}{L} t\right)}$$ $n = 1, 2, 3, \ldots$

for wave, the time solution is also periodic

for each n, there is one solution: $y_n = X_n T_n$

each of these is called a "mode" or "harmonic" general solution:

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a}{L} t\right) \sin\left(\frac{n\pi}{L} x\right)$$

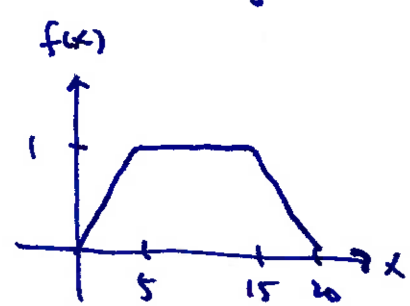
IC: $y(x,0) = f(x)$ initial displacement

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{sine series}$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

example $L = 20$, $a = 1$, $g(x) = 0$ (no initial velocity)

initial displacement $f(x) = \begin{cases} \frac{1}{5}x & 0 < x < 5 \\ 1 & 5 < x < 15 \\ \frac{20-x}{5} & 15 < x < 20 \end{cases}$



$$y(x,t) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}\left[\sin\left(\frac{n\pi}{4}\right) + \sin\left(\frac{3n\pi}{4}\right)\right]\cos\left(\frac{n\pi t}{20}\right)\sin\left(\frac{n\pi x}{20}\right)$$

$n=1$: $y_1(x,t) = \frac{8\sqrt{2}}{\pi^2}\cos\left(\boxed{\frac{\pi}{20}}t\right)\sin\left(\frac{\pi}{20}x\right)$

→ freq. of vibration of the fundamental mode ($n=1$)

$$\frac{\pi}{20}\ \text{rad/s} = \frac{\pi/20}{2\pi} = \frac{1}{40}\ \text{Hz}$$

$n=2$: second harmonic

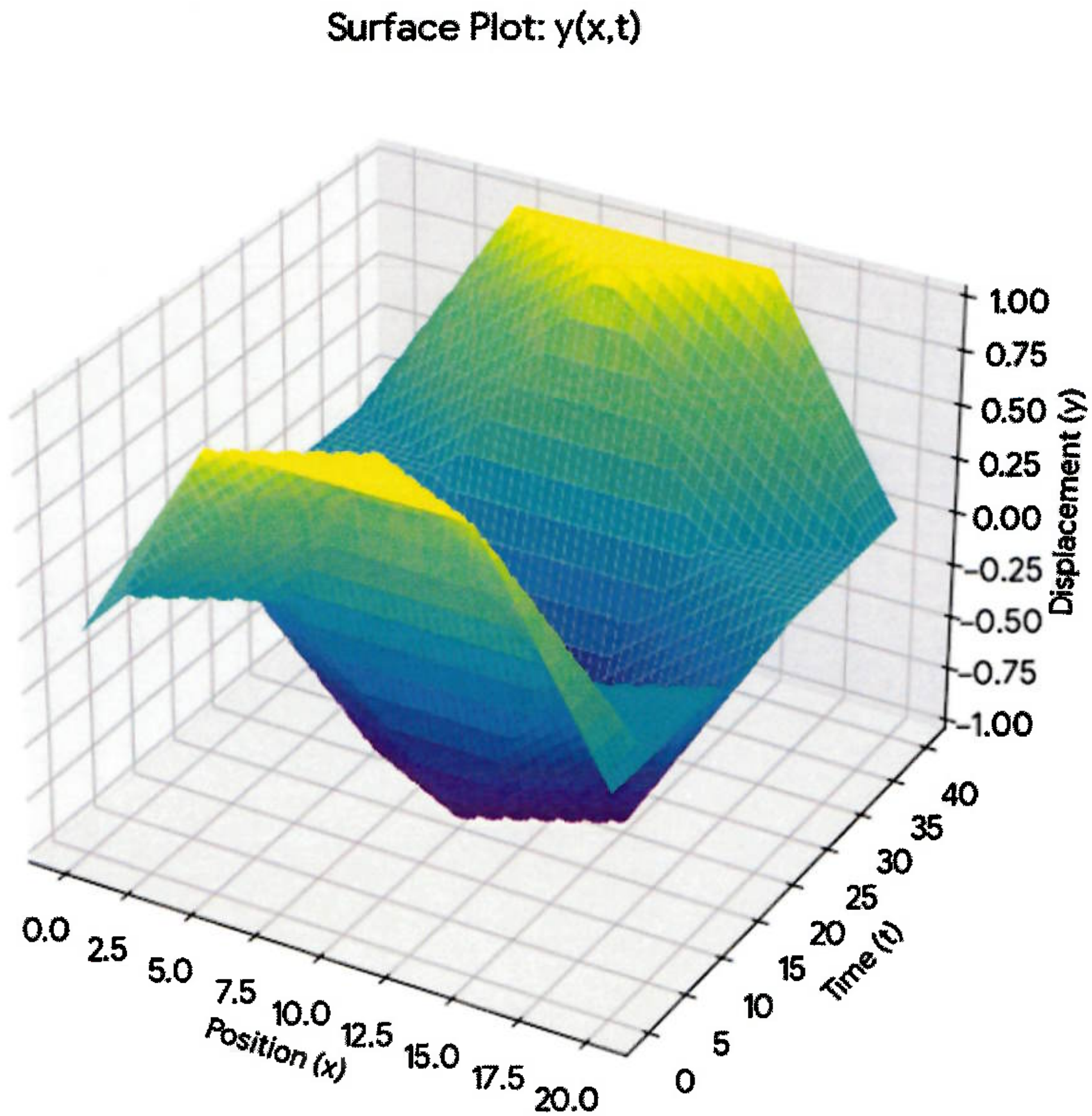
freq. is $2 \cdot \frac{\pi}{20} = \frac{1}{20}$ Hz (double of 1st harmonic)

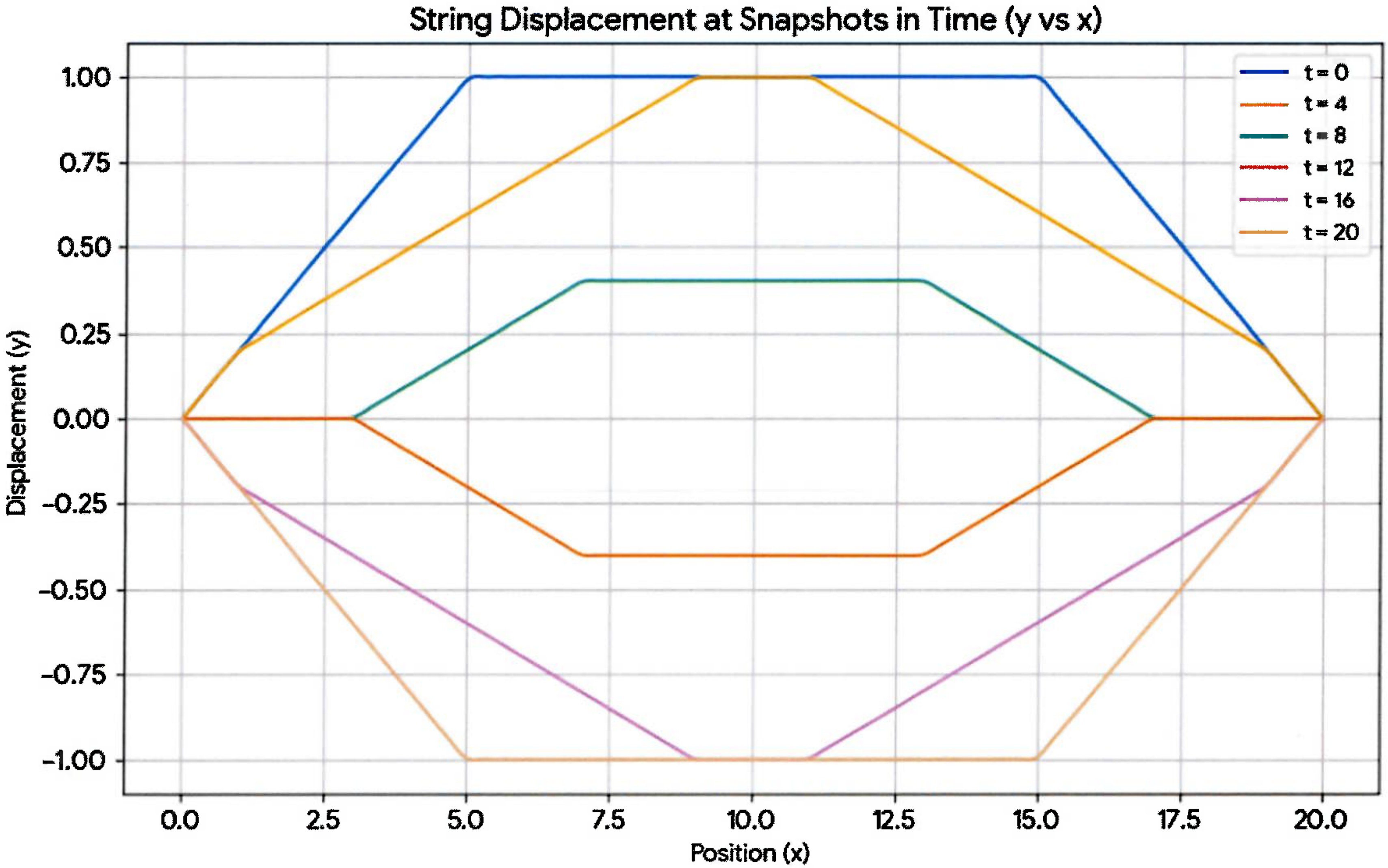
one octave higher than fundamental

$n=3$: third harmonic

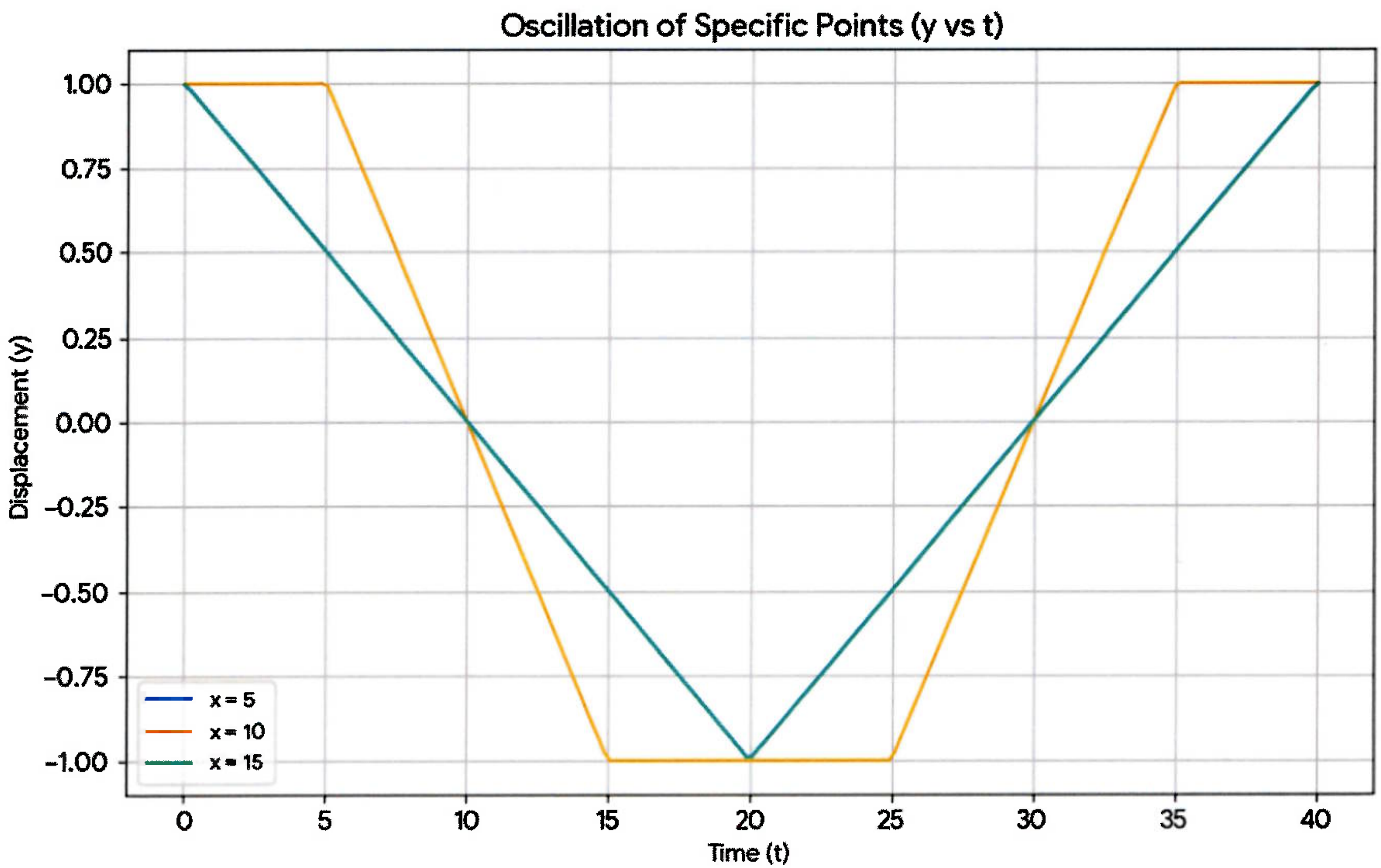
(an octave and a perfect fifth above the fundamental)

the sound we hear is ALL n's put together


Surface Plot: y(x,t)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
Displacement (y)
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
Position (x)
0
5
10
15
20
25
30
35
40
Time (t)



String Displacement at Snapshots in Time (y vs x)
Displacement (y)
Position (x)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
t = 0
t = 4
t = 8
t = 12
t = 16
t = 20


Oscillation of Specific Points (y vs t)
Displacement (y)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
5
10
15
20
25
30
35
40
Time (t)
x = 5
x = 10
x = 15